MATH2050C Assignment 4

Deadline: Feb 5, 2024.

Hand in: 3.2 no. 14b, 16d; 3.3 no. 5, 12c; Suppl Problems no 1, 2, 3.

Section 3.2 no. 14ab, 16bd, 19bd;

Section 3.3 no. 3, 5, 7, 10, 12ac.

Section 3.2

(14b) **Solution** Use Squeeze Theorem in $1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}$ and $\lim_{n\to\infty} n^{1/n} = 1$.

(19d) **Solution** Use
$$\frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\frac{3}{n}\frac{4}{n}\cdots\frac{n}{n} \leq \frac{1}{n}\frac{2}{n}\frac{n}{n}\frac{n}{n}\cdots\frac{n}{n} = \frac{2}{n^2}.$$
By Squeeze Theorem we get
$$\lim_{n\to\infty}\frac{n!}{n^n} = 0 \ .$$

Section 3.3

(5) $y_1 = \sqrt{p}, p > 0$, and $y_{n+1} = \sqrt{p + y_n}$. Use induction it is straightforward to see $\{y_n\}$ is increasing. To show boundedness we follow the hint and use induction to show $y_n \leq 1 + 2\sqrt{p}$. Assuming $y_n \leq 1 + 2\sqrt{p}$, we have
$$y_{n+1}^2 = p + y_n \leq p + 1 + 2\sqrt{p} = (1 + \sqrt{p})^2,$$
hence
$$y_{n+1} \leq 1 + \sqrt{p} < 1 + 2\sqrt{p} \ .$$

(7) It is clear that $x_{n+1} = x_n + 1/x_n, x_1 > 0$, is increasing. Were it bounded from above, its limit exists by Monotone Convergence Theorem. Letting the limit be $b > 0$, then passing limit in the defining relation of the sequence we get $b = b + 1/b$, which is ridiculous. We conclude that $\{x_n\}$ is divergent to infinity.

(10). We claim the sequence $\{y_n\}$ given by
$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$
is increasing and bounded. First, we have
$$y_n < \frac{1}{n} + \frac{1}{n} + \cdots \frac{n}{n} = \frac{n}{n} = 1 \ , \quad \forall n \geq 1,$$
hence $\{y_n\}$ is bounded from above. Next,
$$y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \ .$$
We have
$$y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > 0 \ , \quad \forall n \geq 1 \ ,$$

hence it is increasing. By Monotone Convergence Theorem $\{y_n\}$ is convergent.

Note. One can show that the limit is $\log 2$.

(12)(a) By Limit Theorem

$$\lim_{n\to\infty} = \left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\ .$$

(c) By Limit Theorem

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \frac{\lim_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)} = e\ .$$

Supplementary Problems

1. Suppose that $\lim_{n\to\infty} x_n = x$. Prove that

$$\lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n} = x\ .$$

Solution For $\varepsilon > 0$, fix an n_0 such that $|x_n - x| < \varepsilon/3$ for all $n \ge n_0$. Then fix n_1 such that $|(x_1+\cdots+x_{n_0-1})/n \le \varepsilon/3$ for all $n \ge n_1$ and some n_2 such that $(n_0-1)|x|/n < \varepsilon/3$ for all $n \ge n_2$. Then, for $n \ge \max\{n_0, n_1, n_2\}$,

$$\begin{aligned}\left|\frac{x_1+\cdots+x_n}{n} - x\right| &= \left|\frac{x_1+\cdots+x_{n_0-1}}{n} + \frac{(x_{n_0}-x)+\cdots+(x_n-x)}{n} - \frac{(n_0-1)x}{n}\right| \\ &\le \left|\frac{x_1+\cdots+x_{n_0-1}}{n}\right| + \left|\frac{(x_{n_0}-x)+\cdots+(x_n-x)}{n}\right| + \left|\frac{(n_0-1)|x|}{n}\right| \\ &< \frac{\varepsilon}{3} + \frac{n-n_0+1}{n}\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\ .\end{aligned}$$

2. Determine the limit of

$$\left(1-\frac{a}{n^2}\right)^n\ ,\ a > 0\ .$$

Hint: Use Bernoulli's inequality.

Solution Recall that Bernoulli's inequality $(1+x)^n \ge 1+nx,\ x > -1$. For some large n_0, $-a/n^2 > -1$, and we have $(1-a/n^2)^n \ge 1 - na/n^2 = 1 - a/n$ for all $n \ge n_0$. Therefore, $1 - a/n \le (1-a/n^2)^n \le 1,\ n \ge n_0$, and $\lim_{n\to\infty}(1-a/n^2)^n = 1$ by Squeeze Theorem.

3. Show the limit of $(1-a/n)^n,\ a > 0$ exists. Hint: Use (2).

Solution Using (2),

$$\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^n = \frac{\lim_{n\to\infty}\left(1-a^2/n^2\right)^n}{\lim_{n\to\infty}\left(1+a/n\right)^n} = \frac{1}{E(a)}\ .$$

4. Prove that e is irrational. Hint: Use the inequality $0 < e-(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}) < \frac{1}{k\times k!}$.
Solution Suppose on the contrary that $e = p/q$, a rational number. Then taking $k = q! \geq 2$ in the inequality to get

$$0 < p(q-1)! - q!(1+1+1/2!+\cdots+1/k!) < 1/q \leq 1/2.$$

Noting that $q!(1+1+1/2!+\cdots+1/k!)$ is a natural number, it is impossible to have two distinct natural numbers whose difference is less than $1/2$. Contradiction holds.